206. Proposed by **W. J. GREENSTREET, A. M.,** Editor of The Mathematical Gazette, Stroud. England.

ABCD is circumscribed by a circle center O, and it circumscribes a circle radius r. The perpendiculars from C on the sides are x, y, z, u. Show that $\frac{1}{2}AC.BD=r\Sigma x$.

Solution by **G. B. M. ZERR, A. M., Ph. D.,** Parsons, W. Va.

The problem should read "the perpendiculars from O" instead of "the perpendiculars from C.

Let a, b, c, d denote the sides AB, BC, CD, DA, respectively; x the perpendicular on a; y, on b; z, on c; u, on d; R=circum-radius.

Then $x=\sqrt{(R^2-\frac{1}{4}a^2)}$. Now $R=AC/2\sin B$.

$AC^2=a^2+b^2-2ab\cos B=c^2+d^2+2cd\cos B$.

$$\therefore x=\frac{1}{2\sin B}\sqrt{(AC^2-a^2\sin^2 B)}=\frac{b-a\cos B}{2\sin B},\ y=\frac{a-b\cos B}{2\sin B},$$

$$z=\frac{d+c\cos B}{2\sin B},\ u=\frac{c+d\cos B}{2\sin B}.\quad \therefore r\Sigma x=\frac{r(a+b+c+d)-r(a+b-c-d)\cos B}{2\sin B},$$

$$r=\frac{2\sqrt{(abcd)}}{a+b+c+d},\quad a+c=b+d,\quad \cos B=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}.$$

$$\therefore \cos B=\frac{(a-c)(a+c)+(b-d)(b-d)}{2(ab+cd)}=\frac{(a+c)(a+b-c-d)}{2(ab+cd)}.$$

$$\therefore r\Sigma x=\frac{\sqrt{(abcd)}}{\sin B}-\frac{\sqrt{(abcd)}.\,(a+b-c-d)^2}{4(ab+cd)\sin B},\quad \sin B=\frac{2\sqrt{(abcd)}}{ab+cd},$$

$4(ab+cd)\sin B=8\sqrt{(abcd)}$.

$$\therefore r\Sigma x=\frac{ab+cd}{2}-\frac{(a+b-c-d)^2}{8}.\ \text{But } d=a+c-d.$$

$$\therefore r\Sigma x=\frac{ab+c(a+c-b)}{2}-\frac{(b-c)^2}{2}=\frac{ab+ac+bc-b^2}{2}$$

$$=\frac{ac+b(a+c)-b^2}{2}=\frac{ac+bd}{2}=\tfrac{1}{2}AC.BD.$$

207. Proposed by **W. W. HART,** University High School, Chicago, Ill.

According to Gauss the circumference of a circle can be divided into n equal parts by ruler and compass only, when n is a prime of the form $2^{2^p}+1$.

The following construction gives good partial results for n equals *any* integer. If AB is the diameter of the circle, and C is the vertex of the equilateral triangle ABC, and if D is a point on AB at the distance $2AB/n$ from A, then draw the line CD cutting the circle at E and F; E being the more remote from

C. $AE=1/n$ circumference approximately. For low values of n this method is very practical; is it practical in general? How great is the error?

I. Solution by H. F. MacNEISH, A. B., Instructor in Mathematics, University High School, Chicago, Ill.

Join OE. Let $\angle ACD=x$ and $\angle AOE=y$; then $DCB=60°-x$; $ADC=120°-x$; $DAE=90°-\frac{1}{2}y$; $AED=30°-x+\frac{1}{2}y$. $AD=2AB/n=4r/n$; $AB=AC=BC=2r$; $AO=OE=r$.

In $\triangle ADC$: $$\frac{\sin x}{\sin(120-x)}=\frac{AD}{AC}=\frac{4r/n}{2r}=\frac{2}{n}.$$

$\therefore \frac{1}{2}n\sin x=\sin(120-x)=\frac{1}{2}\sqrt{3}\cos x+\frac{1}{2}\sin x$.....(1). $(n-1)\sin x=\sqrt{3}\sqrt{(1-\sin^2 x)}$.

$$\therefore \sin x=\sqrt{\frac{3}{n^2-2n+4}}\text{.....(2)}. \qquad \therefore \sin(120-x)=\frac{n}{2}\sqrt{\frac{3}{n^2-2n+4}}\text{.....(3)},$$

and $\cos(120-x)=\dfrac{n-4}{2\sqrt{(n^2-2n+4)}}$.....(4).

In $\triangle AOE$: $\dfrac{AE}{OE}=\dfrac{\sin y}{\sin(90-\frac{1}{2}y)}=\dfrac{\sin y}{\cos\frac{1}{2}y}$. $\therefore AE=\dfrac{r\sin y}{\cos\frac{1}{2}y}$.....(5).

In $\triangle AEC$: $\dfrac{AE}{AC}=\dfrac{\sin x}{\sin(30-x+\frac{1}{2}y)}$. $\therefore AE=\dfrac{2r\sin x}{\sin(30-x+\frac{1}{2}y)}$.....(6).

From (5) and (6), $$\frac{\sin y}{\cos\frac{1}{2}y}=\frac{2\sin x}{\sin(30-x+\frac{1}{2}y)}.$$

$$\therefore \sin y[\sin(30-x)\cos\tfrac{1}{2}y+\sin\tfrac{1}{2}y\cos(30-x)]=2\sin x\cos\tfrac{1}{2}y,$$

$$\text{or } \sin y[-\cos(120-x)\cos\tfrac{1}{2}y+\sin\tfrac{1}{2}y\sin(120-x)]=2\sin x\cos\tfrac{1}{2}y.$$

Hence from (1), $\sin y[-\cos(120-x)\cos\frac{1}{2}y+\frac{1}{2}n\sin\frac{1}{2}y\sin x]=2\sin x\cos\frac{1}{2}y$

or $\sin x[n\sin y\sin\frac{1}{2}y-4\cos\frac{1}{2}y]=2\sin y\cos\frac{1}{2}y\cos(120-x)$

or since $\sin y=2\sin\frac{1}{2}y\cos\frac{1}{2}y$, $\sin x[n\sin^2\frac{1}{2}y-2]=\sin y\cos(120-x)$.

Then from (2) and (4),

$$\frac{\sqrt{3}}{\sqrt{(n^2-2n+4)}}[n\sin^2\tfrac{1}{2}y-2]=\frac{n-4}{2\sqrt{(n^2-2n+4)}}\sin y.$$

$$\therefore 2\sqrt{3}\left[\frac{n(1-\cos y)}{2}-2\right]=(n-4)\sin y. \quad \therefore 3(n-n\cos y-4)^2=(1-\cos^2 y)(n-4)^2.$$

$$\therefore \cos y=\frac{(n-4)[3n\pm\sqrt{(n^2+16n-32)}]}{4(n^2-2n+4)}.$$

Then for the positive value of the radical we obtain the following values of y for $n=3, 4, 5, \ldots$

n	$\cos y$	$\log \cos y$	y	$2\pi/n$	Error	Rate of error
3	$-\frac{1}{2}$	——	120°	120°	0	.0000
4	0	——	90°	90°	0	.0000
5	——	9.49107	71° 57′ 12″	72°	2′ 48″	.0007
6	$\frac{1}{2}$	——	60°	60°	0	.0000
7	——	9.79393	51° 31′ 23″	51° 25′ 43″	5′ 40″	.0018
8	——	9.84806	45° 11′ 14″	45°	11′ 14″	.0042
9	——	9.88248	40° 16′ 38″	40°	16′ 38″	.0069
10	——	9.90599	36° 21′ 18″	36°	21′ 18″	.0099
11	——	9.92286	33° 8′ 53″	32° 43′ 38″	25′ 15″	.0129
12	——	9.93545	30° 28′ 15″	30°	28′ 15″	.0157
24	——	9.98363	15° 38′	15°	38′	.0422
48	——	9.99516	8° 32′ 30″	7° 30′	1° 2′ 20″	.1389
90	——	9.99818	5° 14′ 30″	4°	1° 14′ 30″	.3104
180	——	9.99967	2° 15′	2°	15′	.1250
360	——	9.99995	49′ 40″ to 54′ 40″	1°	7′ 30″	.1250

The construction therefore has an error of over $1\frac{1}{2}\%$ for values of $n>12$, and for large values of n the error is very great.

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, West Va.**

Let O be the center of the circle, $AO=R=EO$. Then $CO=R\sqrt{3}$, $DO=R-4R/n=(R/n)(n-4)$. $\therefore$ tangent $DCO=(n-4)/n\sqrt{3}$.

$$\sin DCO=\frac{n-4}{\sqrt{(4n^2-8n+16)}},\quad \cos DCO=\frac{n\sqrt{3}}{\sqrt{(4n^2-8n+16)}},$$

$$\sin CED=\frac{CO\sin DCO}{EO}=\frac{(n-4)\sqrt{3}}{\sqrt{(4n^2-84+16)}}.$$

$$\sin(90^\circ+DOE)=\sin(180^\circ-DEO-DCO)=\sin(DEO+DCO).$$

$$\therefore \cos DOE=\sin(DEO+DCO)=\frac{(n-4)[3n+\sqrt{(n^2+16n-32)}]}{4n^2-8n+16}.$$

For $n=3$, 4 and 6 the error is nothing.
For $n=5$ the side and angle are a trifle small.
For $n>6$ the side and angle are too large but the error varies.
For $n=8$, $\cos DOE=.70479$, $DOE=45°$ 11′ 14.5″, an error of 11′ 14.5″.
For $n=12$, $\cos DOE=.86186$, $DOE=30°$ 28′ 25.7″, an error of 28′ 25.7″.*
For $n=20$, $\cos DOE=.95091$, $DOE=18°$ 2′ 40″, an error of only 2′ 40″.

*In solution I, Mr. MacNeish finds the error for $n=12$ to be 28′15″, otherwise the two solutions agree. Ed.

For $n=72$, $\cos DOE=.99559$, $DOE=5^\circ\,29'$, an error of $29'$.

For large values of n the error is much too great for any purpose.

Also solved by *J. E. SANDERS*, Hackney, Ohio.

CALCULUS.

165. Proposed by CAPT. T. C. DICKSON, Ordnance Department, United States Army, Washington, D. C.

Solve by integration the differential equation

$$\frac{d^2\xi}{dt^2}+\frac{A}{B}\left(\frac{d\xi}{dt}\right)^2-\frac{C}{B}=0,$$

in which A, B, C are given functions of ξ, but independent of t.

Solution by L. E. DICKSON, Ph. D., Assistant Professor of Mathematics, The University of Chicago.

In view of the nature of the coefficients, we regard ξ as the independent variable and t the dependent, the formulae of transformation being

$$\frac{d\xi}{dt}=1\div\frac{dt}{d\xi},\quad \frac{d^2\xi}{dt^2}=-\frac{d^2t}{d\xi^2}\div\left(\frac{dt}{d\xi}\right)^3.$$

The given equation thus becomes

$$\frac{d^2t}{d\xi^2}-\frac{A}{B}\frac{dt}{d\xi}+\frac{C}{B}\left(\frac{dt}{d\xi}\right)^3=0.$$

Set $dt/d\xi=y$, whence $t=\int y d\xi$. Then $\dfrac{dy}{d\xi}-\dfrac{A}{B}y+\dfrac{C}{B}y^3=0.$

Divide by y^3 and set $z=y^{-2}$. The resulting differential equation

$$\frac{dz}{d\xi}+\frac{2A}{B}z-\frac{2C}{B}=0$$

is linear. By the usual method, we get

$$z=2e^{-\lambda}\left(\int\frac{C}{B}e^{\lambda}\,d\xi+k\right),\quad \lambda\equiv 2\int\frac{A}{B}d\xi.\quad \therefore\ t=\int z^{-\frac{1}{2}}d\xi.$$

169. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Find the value of y from the Eulerian equation

$$y=\int\frac{dx}{(x+\sqrt{3})\sqrt[3]{(x^2+1)}}.$$